

Stability of the vanishing of the $\bar{\partial}_b$ -cohomology under small horizontal perturbations of the CR structure in compact abstract q -concave CR manifolds

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The tangential Cauchy-Riemann equation is one of the main tools in CR analysis and its properties are deeply related to the geometry of CR manifolds, in particular the complex tangential directions are playing an important role. For example it was noticed by Folland and Stein [3], when they studied the tangential Cauchy-Riemann operator on the Heisenberg group and more generally on strictly pseudoconvex real hypersurfaces of \mathbb{C}^n , that one get better estimates in the complex tangential directions. Therefore in the study of the stability properties for the tangential Cauchy-Riemann equation under perturbations of the CR structure it seems natural to consider perturbations which preserve the complex tangent vector bundle. Such perturbations can be represented as graphs in the complex tangent vector bundle over the original CR structure, they are defined by $(0, 1)$ -forms with values in the holomorphic tangent bundle. We call them horizontal perturbations.

We consider compact abstract CR manifolds and integrable perturbations of their CR structure which preserve their complex tangent vector bundle. Since the Levi form of a CR manifold depends only on its complex tangent vector bundle, such perturbations will preserve the Levi form and hence the concavity properties of the manifold which are closely related to the $\bar{\partial}_b$ -cohomology. For example it is well known that, for q -concave compact CR manifolds of real dimension $2n - k$ and CR-dimension $n - k$, the $\bar{\partial}_b$ -cohomology groups are finite dimensional in bidegree (p, r) , when $1 \leq p \leq n$ and $1 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$. Therefore, for a q -concave compact CR manifold, this finiteness property is stable by horizontal perturbations of the CR structure.

In this paper we are interested in the stability of the vanishing of the $\bar{\partial}_b$ -cohomology groups after horizontal perturbations of the CR structure.

Let $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class \mathcal{C}^∞ , of real dimension $2n - k$ and CR dimension $n - k$, and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ another abstract compact CR manifold such that $\widehat{H}_{0,1}\mathbb{M}$ is a smooth integrable horizontal perturbation of $H_{0,1}\mathbb{M}$, then $\widehat{H}_{0,1}\mathbb{M}$ is defined by a smooth form $\Phi \in \mathcal{C}_{0,1}^\infty(\mathbb{M}, H_{1,0}\mathbb{M})$. We denote by $\bar{\partial}_b$ the tangential Cauchy-Riemann operator associated to the CR structure $H_{0,1}\mathbb{M}$ and by $\bar{\partial}_b^\Phi$ the tangential Cauchy-Riemann operator associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$.

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The smooth $\bar{\partial}_b$ -cohomology groups on \mathbb{M} in bidegree $(0, r)$ and (n, r) are defined for $1 \leq r \leq n - k$ by:

$$H^{0,r}(\mathbb{M}) = \{f \in \mathcal{C}_{0,r}^\infty(\mathbb{M}) \mid \bar{\partial}_b f = 0\} / \bar{\partial}_b(\mathcal{C}_{0,r-1}^\infty(\mathbb{M}))$$

and

$$H^{n,r}(\mathbb{M}) = \{f \in \mathcal{C}_{n,r}^\infty(\mathbb{M}) \mid \bar{\partial}_b f = 0\} / \bar{\partial}_b(\mathcal{C}_{n,r-1}^\infty(\mathbb{M})).$$

If f is a smooth differential form of degree r , $1 \leq r \leq n - k$, on \mathbb{M} , we denote by $f_{r,0}$ its projection on the space $\mathcal{C}_{r,0}^\infty(\mathbb{M})$ of $(r, 0)$ -forms for the CR structure $H_{0,1}\mathbb{M}$. Note that if $r \geq n + 1$ then $f_{r,0} = 0$.

The smooth $\bar{\partial}_b^\Phi$ -cohomology groups on $\hat{\mathbb{M}}$ in bidegree $(0, r)$ and (n, r) are defined for $1 \leq r \leq n - k$ by:

$$H_\Phi^{0,r}(\hat{\mathbb{M}}) = \{f \in \mathcal{C}_{0,r}^\infty(\hat{\mathbb{M}}) \mid f_{r,0} = 0, \bar{\partial}_b^\Phi f = 0\} / \bar{\partial}_b^\Phi(\mathcal{C}_{0,r-1}^\infty(\hat{\mathbb{M}}))$$

and

$$H_\Phi^{n,r}(\hat{\mathbb{M}}) = \{f \in \mathcal{C}_{n,r}^\infty(\hat{\mathbb{M}}) \mid \bar{\partial}_b^\Phi f = 0\} / \bar{\partial}_b^\Phi(\mathcal{C}_{n,r-1}^\infty(\hat{\mathbb{M}})).$$

In this paper the following stability result is proved:

Theorem 0.1. *Assume \mathbb{M} is q -concave, there exists then a sequence $(\delta_l)_{l \in \mathbb{N}}$ of positive real numbers such that, if $\|\Phi\|_l < \delta_l$ for each $l \in \mathbb{N}$,*

(i) $H^{p,r-p}(\mathbb{M}) = 0$ for all $1 \leq p \leq r$ implies $H_\Phi^{0,r}(\hat{\mathbb{M}}) = 0$ when $1 \leq r \leq q - 2$, in the abstract case, and when $1 \leq r \leq q - 1$, if \mathbb{M} is locally embeddable,

(ii) $H^{n-p,r+p}(\mathbb{M}) = 0$ for all $0 \leq p \leq n - k - r$ implies $H_\Phi^{n,r}(\hat{\mathbb{M}}) = 0$ when $n - k - q + 1 \leq r \leq n - k$.

We also prove the stability of the solvability of the tangential Cauchy-Riemann equation with sharp anisotropic regularity (cf. Theorem 3.3).

Note that, when both CR manifolds \mathbb{M} and $\hat{\mathbb{M}}$ are embeddable in the same complex manifold (i.e. in the embedded case), a $(0, r)$ -form f for the new CR structure $\hat{H}_{0,1}\mathbb{M}$ is also a $(0, r)$ -form for the original CR structure $H_{0,1}\mathbb{M}$ and hence the condition $f_{r,0} = 0$ in the definition of the cohomology groups $H_\Phi^{0,r}(\hat{\mathbb{M}})$ is automatically fulfilled. In that case, Polyakov proved in [7] global homotopy formulas for a family of CR manifolds in small degrees which immediately imply the stability of the vanishing of the $\bar{\partial}_b$ -cohomology groups of small degrees. In his paper he does not need the perturbation to preserve the complex tangent vector bundle, but his estimates are far to be sharp.

Moreover Polyakov [8] proved also that if a generically embedded compact CR manifold $M \subset X$ is at least 3-concave and satisfies $H^{0,1}(M, T'X|_M) = 0$, then small perturbations of the CR structure are still embeddable in the same manifold X . From this result and the global homotopy formula from [7] one can derive some stability results for the vanishing of $\bar{\partial}_b$ -cohomology groups of small degrees without hypothesis on the embeddability a priori of the perturbed CR structure.

The main interest of our paper is that we do not assume the CR manifolds \mathbb{M} and $\hat{\mathbb{M}}$ to be embeddable. In the case where the original CR manifold \mathbb{M} is embeddable it covers the case where it is unknown if the perturbed CR structure is embeddable in the same

manifold as the original one, for example when the manifold \mathbb{M} is only 2-concave. Finally we also reach the case of the $\bar{\partial}_b$ -cohomology groups of large degrees, which, even in the embedded case, cannot be deduced from the works of Polyakov.

The main tool in the proof of the stability of the vanishing of the $\bar{\partial}_b$ -cohomology groups is a fixed point theorem which is derived from global homotopy formulas with sharp anisotropic estimates. Such formulas are proved in [9], in the abstract case, by using the L^2 theory for the \square_b operator and in [5], in the locally embeddable case, by first proving that the integral operators associated to the kernels built in [1] satisfies sharp anisotropic estimates, which implies local homotopy formulas with sharp anisotropic estimates, and then by using the globalization method from [6] and [2].

1 CR Structures

Let \mathbb{M} be a \mathcal{C}^l -smooth, $l \geq 1$, paracompact differential manifold, we denote by $T\mathbb{M}$ the tangent bundle of \mathbb{M} and by $T_{\mathbb{C}}\mathbb{M} = \mathbb{C} \otimes T\mathbb{M}$ the complexified tangent bundle.

Definition 1.1. An *almost CR structure* on \mathbb{M} is a subbundle $H_{0,1}\mathbb{M}$ of $T_{\mathbb{C}}\mathbb{M}$ such that $H_{0,1}\mathbb{M} \cap \overline{H_{0,1}\mathbb{M}} = \{0\}$.

If the almost CR structure is integrable, i.e. for all $Z, W \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ then $[Z, W] \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$, then it is called a *CR structure*.

If $H_{0,1}\mathbb{M}$ is a CR structure, the pair $(\mathbb{M}, H_{0,1}\mathbb{M})$ is called an *abstract CR manifold*.

The *CR dimension* of \mathbb{M} is defined by $\text{CR-dim } \mathbb{M} = \text{rk}_{\mathbb{C}} H_{0,1}\mathbb{M}$.

We set $H_{1,0}\mathbb{M} = \overline{H_{0,1}\mathbb{M}}$ and we denote by $H^{0,1}\mathbb{M}$ the dual bundle $(H_{0,1}\mathbb{M})^*$ of $H_{0,1}\mathbb{M}$.

Let $\Lambda^{0,q}\mathbb{M} = \bigwedge^q(H^{0,1}\mathbb{M})$, then $\mathcal{C}_{0,q}^s(\mathbb{M}) = \Gamma^s(\mathbb{M}, \Lambda^{0,q}\mathbb{M})$ is called the space of $(0, q)$ -forms of class \mathcal{C}^s , $0 \leq s \leq l$, on \mathbb{M} .

We define $\Lambda^{p,0}\mathbb{M}$ as the space of forms of degree p that annihilate any p -vector on \mathbb{M} that has more than one factor contained in $H_{0,1}\mathbb{M}$. Then $\mathcal{C}_{p,q}^s(\mathbb{M}) = \mathcal{C}_{0,q}^s(\mathbb{M}, \Lambda^{p,0}\mathbb{M})$ is the space of $(0, q)$ -forms of class \mathcal{C}^s with values in $\Lambda^{p,0}\mathbb{M}$.

If the almost CR structure is a CR structure, i.e. if it is integrable, and if $s \geq 1$, then we can define an operator

$$\bar{\partial}_b : \mathcal{C}_{0,q}^s(\mathbb{M}) \rightarrow \mathcal{C}_{0,q+1}^{s-1}(\mathbb{M}), \quad (1.1)$$

called the *tangential Cauchy-Riemann operator*, by setting $\bar{\partial}_b f = df|_{H_{0,1}\mathbb{M} \times \dots \times H_{0,1}\mathbb{M}}$. It satisfies $\bar{\partial}_b \circ \bar{\partial}_b = 0$.

Definition 1.2. Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract CR manifold, X be a complex manifold and $F : \mathbb{M} \rightarrow X$ be an embedding of class \mathcal{C}^l , then F is called a *CR embedding* if $dF(H_{0,1}\mathbb{M})$ is a subbundle of the bundle $T_{0,1}X$ of the antiholomorphic vector fields of X and $dF(H_{0,1}\mathbb{M}) = T_{0,1}X \cap T_{\mathbb{C}}F(\mathbb{M})$.

Let F be a CR embedding of an abstract CR manifold \mathbb{M} into a complex manifold X and set $M = F(\mathbb{M})$, then M is a CR manifold with the CR structure $H_{0,1}M = T_{0,1}X \cap T_{\mathbb{C}}M$.

Let U be a coordinate domain in X , then $F|_{F^{-1}(U)} = (f_1, \dots, f_N)$, with $N = \dim_{\mathbb{C}} X$, and F is a CR embedding if and only if, for all $1 \leq j \leq N$, $\bar{\partial}_b f_j = 0$.

A CR embedding is called *generic* if $\dim_{\mathbb{C}} X - \text{rk}_{\mathbb{C}} H_{0,1}\mathbb{M} = \text{codim}_{\mathbb{R}} M$.

Definition 1.3. An almost CR structure $\hat{H}_{0,1}\mathbb{M}$ on \mathbb{M} is said to be of *finite distance* to a given CR structure $H_{0,1}\mathbb{M}$ if $\hat{H}_{0,1}\mathbb{M}$ can be represented as a graph in $T_{\mathbb{C}}\mathbb{M}$ over $H_{0,1}\mathbb{M}$.

It is called an *horizontal perturbation* of the CR structure $H_{0,1}\mathbb{M}$ if it is of finite distance to $H_{0,1}\mathbb{M}$ and moreover there exists $\Phi \in \mathcal{C}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$ such that

$$\hat{H}_{0,1}\mathbb{M} = \{\overline{W} \in T_{\mathbb{C}}\mathbb{M} \mid \overline{W} = \overline{Z} - \Phi(\overline{Z}), \overline{Z} \in H_{0,1}\mathbb{M}\}, \quad (1.2)$$

which means that $\hat{H}_{0,1}\mathbb{M}$ is a graph in $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$.

Note that an horizontal perturbation of the original CR structure preserves the complex tangent bundle $H\mathbb{M}$.

Assume \mathbb{M} is an abstract CR manifold and $\hat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of the original CR structure $H_{0,1}\mathbb{M}$ on \mathbb{M} . If $\overline{\partial}_b^\Phi$ denotes the tangential Cauchy-Riemann operator associated to the CR structure $\hat{H}_{0,1}\mathbb{M}$, then we have

$$\overline{\partial}_b^\Phi = \overline{\partial}_b - \Phi \lrcorner d = \overline{\partial}_b - \Phi \lrcorner \partial_b, \quad (1.3)$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator associated to the original CR structure $H_{0,1}\mathbb{M}$ and ∂_b involves only holomorphic tangent vector fields.

The annihilator $H^0\mathbb{M}$ of $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ in $T_{\mathbb{C}}^*\mathbb{M}$ is called the *characteristic bundle* of \mathbb{M} . Given $p \in \mathbb{M}$, $\omega \in H_p^0\mathbb{M}$ and $X, Y \in H_p\mathbb{M}$, we choose $\tilde{\omega} \in \Gamma(\mathbb{M}, H^0\mathbb{M})$ and $\tilde{X}, \tilde{Y} \in \Gamma(\mathbb{M}, H\mathbb{M})$ with $\tilde{\omega}_p = \omega$, $\tilde{X}_p = X$ and $\tilde{Y}_p = Y$. Then $d\tilde{\omega}(X, Y) = -\omega([\tilde{X}, \tilde{Y}])$. Therefore we can associate to each $\omega \in H_p^0\mathbb{M}$ an hermitian form

$$L_\omega(X) = -i\omega([\tilde{X}, \overline{\tilde{X}}]) \quad (1.4)$$

on $H_p\mathbb{M}$. This is called the *Levi form* of \mathbb{M} at $\omega \in H_p^0\mathbb{M}$.

In the study of the $\overline{\partial}_b$ -complex two important geometric conditions were introduced for CR manifolds of real dimension $2n - k$ and CR-dimension $n - k$. The first one by Kohn in the hypersurface case, $k = 1$, the condition $Y(q)$, the second one by Henkin in codimension k , $k \geq 1$, the q -concavity.

An abstract CR manifold \mathbb{M} of hypersurface type satisfies Kohn's condition $Y(q)$ at a point $p \in \mathbb{M}$ for some $0 \leq q \leq n - 1$, if the Levi form of \mathbb{M} at p has at least $\max(n - q, q + 1)$ eigenvalues of the same sign or at least $\min(n - q, q + 1)$ eigenvalues of opposite signs.

An abstract CR manifold \mathbb{M} is said to be q -concave at $p \in \mathbb{M}$ for some $0 \leq q \leq n - k$, if the Levi form L_ω at $\omega \in H_p^0\mathbb{M}$ has at least q negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H_p^0\mathbb{M}$.

In [9] the condition $Y(q)$ is extended to arbitrary codimension.

Definition 1.4. An abstract CR manifold is said to satisfy *condition $Y(q)$* for some $1 \leq q \leq n - k$ at $p \in \mathbb{M}$ if the Levi form L_ω at $\omega \in H_p^0\mathbb{M}$ has at least $n - k - q + 1$ positive eigenvalues or at least $q + 1$ negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H_p^0\mathbb{M}$.

Note that in the hypersurface case, i.e. $k = 1$, this condition is equivalent to the classical condition $Y(q)$ of Kohn for hypersurfaces. Moreover, if \mathbb{M} is q -concave at $p \in \mathbb{M}$, then $q \leq (n - k)/2$ and condition $Y(r)$ is satisfied at $p \in \mathbb{M}$ for any $0 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$.

2 Stability of vanishing theorems by horizontal perturbations of the CR structure

Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class \mathcal{C}^∞ , of real dimension $2n - k$ and CR dimension $n - k$, and $\widehat{H}_{0,1}\mathbb{M}$ be an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$. We denote by \mathbb{M} the abstract CR manifold $(\mathbb{M}, H_{0,1}\mathbb{M})$ and by $\widehat{\mathbb{M}}$ the abstract CR manifold $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$.

Since $\widehat{H}_{0,1}\mathbb{M}$ is an horizontal perturbation of $H_{0,1}\mathbb{M}$, which means that $\widehat{H}_{0,1}\mathbb{M}$ is a graph in $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$, the space $H\widehat{\mathbb{M}} = H_{1,0}\widehat{\mathbb{M}} \oplus H_{0,1}\widehat{\mathbb{M}}$ coincides with the space $H\mathbb{M}$ and consequently the two abstract CR manifolds \mathbb{M} and $\widehat{\mathbb{M}}$ have the same characteristic bundle and hence the same Levi form. This implies in particular that if \mathbb{M} satisfies condition Y(q) at each point, then $\widehat{\mathbb{M}}$ satisfies also condition Y(q) at each point.

It follows from the Hodge decomposition theorem and the results in [9] that if \mathbb{M} is an abstract compact CR manifold of class \mathcal{C}^∞ which satisfies condition Y(q) at each point, then the cohomology groups $H^{p,q}(\mathbb{M})$, $0 \leq p \leq n$, are finite dimensional. A natural question is then the stability by small horizontal perturbations of the CR structure of the vanishing of these groups.

Let us consider a sequence $(\mathcal{B}^l(\mathbb{M}), l \in \mathbb{N})$ of Banach spaces with $\mathcal{B}^{l+1}(\mathbb{M}) \subset \mathcal{B}^l(\mathbb{M})$, which are invariant by horizontal perturbations of the CR structure of \mathbb{M} and such that if $f \in \mathcal{B}^l(\mathbb{M})$, $l \geq 1$, then $X_{\mathbb{C}}f \in \mathcal{B}^{l-1}(\mathbb{M})$ for all complex tangent vector fields $X_{\mathbb{C}}$ to \mathbb{M} and there exists $\theta(l) \in \mathbb{N}$ with $\theta(l+1) \geq \theta(l)$ such that $fg \in \mathcal{B}^l(\mathbb{M})$ if $f \in \mathcal{B}^l(\mathbb{M})$ and $g \in \mathcal{C}^{\theta(l)}(\mathbb{M})$. Such a sequence $(\mathcal{B}^l(\mathbb{M}), l \in \mathbb{N})$ will be called a *sequence of anisotropic spaces*. We denote by $\mathcal{B}_{p,r}^l(\mathbb{M})$ the space of (p, r) -forms on \mathbb{M} whose coefficients belong to $\mathcal{B}^l(\mathbb{M})$. Moreover we will say that these Banach spaces are *adapted to the $\bar{\partial}_b$ -equation in degree $r \geq 1$* if, when $H^{p,r}(\mathbb{M}) = 0$, $0 \leq p \leq n$, there exist linear continuous operators A_s , $s = r, r+1$, from $\mathcal{B}_{p,s}^0(\mathbb{M})$ into $\mathcal{B}_{p,s-1}^0(\mathbb{M})$ which are also continuous from $\mathcal{B}_{p,s}^l(\mathbb{M})$ into $\mathcal{B}_{p,s-1}^{l+1}(\mathbb{M})$, $l \in \mathbb{N}$, and moreover satisfy

$$f = \bar{\partial}_b A_r f + A_{r+1} \bar{\partial}_b f, \quad (2.1)$$

for $f \in \mathcal{B}_{p,r}^l(\mathbb{M})$ with $\bar{\partial}_b f = 0$.

Theorem 2.1. *Let $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class \mathcal{C}^∞ , of real dimension $2n - k$ and CR dimension $n - k$, and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ another abstract compact CR manifold such that $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$. Let also $(\mathcal{B}^l(\mathbb{M}), l \in \mathbb{N})$ be a sequence of anisotropic Banach spaces and q be an integer, $1 \leq q \leq (n - k)/2$. Finally let $\Phi \in \mathcal{C}_{0,1}^{\theta(l)}(\mathbb{M}, H_{1,0}\mathbb{M})$ be the differential form which defines the tangential Cauchy-Riemann operator $\bar{\partial}_b^\Phi = \bar{\partial}_b - \Phi \lrcorner \bar{\partial}_b$ associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$.*

Assume $H^{p,r-p}(\mathbb{M}) = 0$, for $1 \leq p \leq r$ and $1 \leq r \leq s_1(q)$, or $H^{n-p,r+p}(\mathbb{M}) = 0$, for $0 \leq p \leq n - k - r$ and $s_2(q) \leq r \leq n - k$ and that the Banach spaces $(\mathcal{B}^l(\mathbb{M}), l \in \mathbb{N})$ are adapted to the $\bar{\partial}_b$ -equation in degree r , $1 \leq r \leq s_1(q)$ or $s_2(q) \leq r \leq n - k$. Then, for each $l \in \mathbb{N}$, there exists $\delta > 0$ such that, if $\|\Phi\|_{\theta(l)} < \delta$,

- (i) for each $\bar{\partial}_b^\phi$ -closed form f in $\mathcal{B}_{0,r}^l(\widehat{\mathbb{M}})$, $1 \leq r \leq s_1(q)$, such that the part of f of bidegree $(0, r)$ for the initial CR structure $H_{0,1}\mathbb{M}$ vanishes, there exists a form u in $\mathcal{B}_{0,r-1}^{l+1}(\widehat{\mathbb{M}})$ satisfying $\bar{\partial}_b^\phi u = f$,
- (ii) for each $\bar{\partial}_b^\phi$ -closed form f in $\mathcal{B}_{n,r}^l(\widehat{\mathbb{M}})$, $s_2(q) \leq r \leq n - k$, there exists a form u in $\mathcal{B}_{n,r-1}^{l+1}(\widehat{\mathbb{M}})$ satisfying $\bar{\partial}_b^\phi u = f$.

Remark 2.2. Note that if both $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ are embeddable in the same complex manifold X , any r -form on the differential manifold \mathbb{M} , which represents a form of bidegree $(0, r)$ for the CR structure $\widehat{H}_{0,1}\mathbb{M}$ represents also a form of bidegree $(0, r)$ for the CR structure $H_{0,1}\mathbb{M}$. Hence the bidegree hypothesis in (i) of Theorem 2.1 is automatically fulfilled.

Proof. Let $f \in \mathcal{B}_{0,r}^l(\widehat{\mathbb{M}})$ be a $(0, r)$ -form for the CR structure $\widehat{H}_{0,1}\mathbb{M}$, $1 \leq r \leq s_1(q)$, such that $\bar{\partial}_b^\Phi f = 0$, we want to solve the equation

$$\bar{\partial}_b^\Phi u = f. \quad (2.2)$$

The form f can be written $\sum_{p=0}^r f_{p,r-p}$, where the forms $f_{p,r-p}$ are of type $(p, r-p)$ for the CR structure $H_{0,1}\mathbb{M}$. Then by considerations of bidegrees, the equation $\bar{\partial}_b^\Phi f = 0$ is equivalent to the family of equations $\bar{\partial}_b^\Phi f_{p,r-p} = 0$, $0 \leq p \leq r$.

Moreover, if $u = \sum_{s=0}^{r-1} u_{s,r-1-s}$, where the forms $u_{s,r-1-s}$ are of type $(s, r-1-s)$ for the CR structure $H_{0,1}\mathbb{M}$, is a solution of (2.2), then

$$\bar{\partial}_b^\Phi u_{p,r-1-p} = f_{p,r-p},$$

for $0 \leq p \leq r-1$, and

$$f_{r,0} = 0.$$

Therefore a necessary condition on f for the solvability of (2.2) is that $\bar{\partial}_b^\Phi f = 0$ and $f_{r,0} = 0$, where $f_{r,0}$ is the part of type $(r, 0)$ of f for the CR structure $H_{0,1}\mathbb{M}$, and, to solve (2.2), we have to consider the equation

$$\bar{\partial}_b^\Phi v = g, \quad (2.3)$$

where $g \in \mathcal{B}_{p,r-p}^l(\mathbb{M})$ is a $(p, r-p)$ -form for the CR structure $H_{0,1}\mathbb{M}$, $0 \leq p \leq r-1$, which is $\bar{\partial}_b^\Phi$ closed. By definition of the operator $\bar{\partial}_b^\Phi$, this means solving the equation $\bar{\partial}_b v = g + \Phi \lrcorner \partial_b v$. Consequently if v is a solution of (2.3), then $\bar{\partial}_b(g + \Phi \lrcorner \partial_b v) = 0$ and by (2.1)

$$\bar{\partial}_b(A_{r-p}(g + \Phi \lrcorner \partial_b v)) = g + \Phi \lrcorner \partial_b v.$$

Assume Φ is of class $\mathcal{C}^{\theta(l)}$, then the map

$$\begin{aligned} \Theta : \mathcal{B}_{p,r-1}^{l+1}(\mathbb{M}) &\rightarrow \mathcal{B}_{p,r-1}^{l+1}(\mathbb{M}) \\ v &\mapsto A_{r-p}g + A_{r-p}(\Phi \lrcorner \partial_b v). \end{aligned}$$

is continuous, and the fixed points of Θ are good candidates to be solutions of (2.3).

Let δ_0 such that, if $\|\Phi\|_{\theta(l)} < \delta_0$, then the norm of the bounded endomorphism $A_{r-p} \circ \Phi \lrcorner \partial_b$ of $\mathcal{B}_{p,r-p-1}^{l+1}(\mathbb{M})$ is equal to $\epsilon_0 < 1$. We shall prove that, if $\|\Phi\|_{\theta(l)} < \delta_0$, Θ admits a unique fixed point.

Consider first the uniqueness of the fixed point. Assume v_1 and v_2 are two fixed points of Θ , then

$$\begin{aligned} v_1 &= \Theta(v_1) = A_{r-p}g + A_{r-p}(\Phi \lrcorner \partial_b v_1) \\ v_2 &= \Theta(v_2) = A_{r-p}g + A_{r-p}(\Phi \lrcorner \partial_b v_2). \end{aligned}$$

This implies

$$v_1 - v_2 = A_{r-p}(\Phi \lrcorner \partial_b(v_1 - v_2))$$

and, by the hypothesis on Φ ,

$$\|v_1 - v_2\|_{\mathcal{B}^{l+1}} < \|v_1 - v_2\|_{\mathcal{B}^{l+1}} \quad \text{or} \quad v_1 = v_2$$

and hence $v_1 = v_2$.

For the existence we proceed by iteration. We set $v_0 = \Theta(0) = A_{r-p}(g)$ and, for $n \geq 0$, $v_{n+1} = \Theta(v_n)$. Then for $n \geq 0$, we get

$$v_{n+1} - v_n = A_{r-p}(\Phi \lrcorner \partial_b(v_n - v_{n-1})).$$

Therefore, if $\|\Phi\|_{\theta(l)} < \delta_0$, the sequence $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{B}_{p,r-1}^{l+1}(\mathbb{M})$ and hence converges to a form v , moreover by continuity of the map Θ , v satisfies $\Theta(v) = v$.

It remains to prove that v is a solution of (2.3). Since $H^{p,r-p}(\mathbb{M}) = 0$ for $1 \leq p \leq r$, it follows from (2.1) and from the definition of the sequence $(v_n)_{n \in \mathbb{N}}$ that

$$g - \bar{\partial}_b^\Phi v_{n+1} = \Phi \lrcorner \partial_b(v_{n+1} - v_n) + A_{r-p+1} \bar{\partial}_b(g + \Phi \lrcorner \partial_b v_n)$$

and since

$$\begin{aligned} \bar{\partial}_b(g + \Phi \lrcorner \partial_b v_n) &= \bar{\partial}_b g - \bar{\partial}_b(\bar{\partial}_b - \Phi \lrcorner \partial_b)v_n \\ &= \bar{\partial}_b g - \bar{\partial}_b(\bar{\partial}_b^\Phi v_n) \\ &= \bar{\partial}_b g - (\bar{\partial}_b^\Phi + \Phi \lrcorner \partial_b)(\bar{\partial}_b^\Phi v_n) \\ &= \bar{\partial}_b g - \Phi \lrcorner \partial_b(\bar{\partial}_b^\Phi v_n), \quad \text{since } (\bar{\partial}_b^\Phi)^2 = 0 \\ &= \Phi \lrcorner \partial_b(g - \bar{\partial}_b^\Phi v_n), \quad \text{since } \bar{\partial}_b^\Phi g = 0, \end{aligned}$$

we get

$$g - \bar{\partial}_b^\Phi v_{n+1} = \Phi \lrcorner \partial_b(v_{n+1} - v_n) + A_{r-p+1}(\Phi \lrcorner \partial_b(g - \bar{\partial}_b^\Phi v_n)). \quad (2.4)$$

Note that since $g \in \mathcal{B}_{p,r}^l(\mathbb{M})$ and Φ is of class $\mathcal{C}^{\theta(l)}$, it follows from the definition of the v_n s that $v_n \in \mathcal{B}_{p,r}^{l+1}(\mathbb{M})$ and $\bar{\partial}_b^\Phi v_n \in \mathcal{B}_{p,r}^l(\mathbb{M})$ for all $n \in \mathbb{N}$.

Thus by (2.4), we have the estimate

$$\|g - \bar{\partial}_b^\Phi v_{n+1}\|_{\mathcal{B}^l} \leq \|\Phi \lrcorner \partial_b\| \|(v_{n+1} - v_n)\|_{\mathcal{B}^{l+1}} + \|A_{r-p+1} \circ \Phi \lrcorner \partial_b\| \|g - \bar{\partial}_b^\Phi v_n\|_{\mathcal{B}^l}. \quad (2.5)$$

Let δ such that if $\|\Phi\|_{\theta(l)} < \delta$, then the maximum of the norm of the bounded endomorphisms $A_s \circ \Phi \lrcorner \partial_b$, $s = r - p, r - p + 1$, of $\mathcal{B}_{p,s-1}^l(\mathbb{M})$ is equal to $\epsilon < 1$. Assume $\|\Phi\|_{\theta(l)} < \delta$, then by induction we get

$$\|g - \bar{\partial}_b^\Phi v_{n+1}\|_{\mathcal{B}^l} \leq (n+1)\epsilon^{n+1} \|\Phi \lrcorner \partial_b\| \|v_0\|_{\mathcal{B}^{l+1}} + \epsilon^{n+1} \|g - \bar{\partial}_b^\Phi v_0\|_{\mathcal{B}^l}. \quad (2.6)$$

But $g - \bar{\partial}_b^\Phi v_0 = \Phi \lrcorner \partial_b A_{r-p} g + A_{r-p+1}(\Phi \lrcorner \partial_b g)$ and hence $\|g - \bar{\partial}_b^\Phi v_0\|_{\mathcal{B}^l} \leq \|\Phi \lrcorner \partial_b\| \|A_{r-p} g\|_{\mathcal{B}^{l+1}} + \epsilon \|g\|_{\mathcal{B}^l}$. This implies

$$\|g - \bar{\partial}_b^\Phi v_{n+1}\|_{\mathcal{B}^l} \leq (n+2)\epsilon^{n+1} \|\Phi \lrcorner \partial_b\| \|A_{r-p}\| \|g\|_{\mathcal{B}^l} + \epsilon^{n+2} \|g\|_{\mathcal{B}^l}. \quad (2.7)$$

Since $\epsilon < 1$, the righthand side of (2.7) tends to zero, when n tends to infinity and by continuity of the operator $\bar{\partial}_b^\Phi$ from $\mathcal{B}_{p,r-1}^{l+1}(\mathbb{M})$ into $\mathcal{B}_{p,r}^l(\mathbb{M})$, the lefthand side of (2.7) tends to $\|g - \bar{\partial}_b^\Phi v\|_{\mathcal{B}^l}$, when n tends to infinity, which implies that v is a solution of (2.3).

Now if $f \in \mathcal{B}_{n,r}^l(\widehat{\mathbb{M}})$ is an (n, r) -form for the CR structure $\widehat{H}_{0,1}\mathbb{M}$, $s_2(q) \leq r \leq n - k$, such that $\bar{\partial}_b^\Phi f = 0$, then the form f can be written $\sum_{p=0}^{n-k-r} f_{n-p,r+p}$, where the forms $f_{n-p,r+p}$ are $\bar{\partial}_b^\Phi$ -closed and of type $(n - p, r + p)$ for the CR structure $H_{0,1}\mathbb{M}$. Then to solve the equation

$$\bar{\partial}_b^\Phi u = f,$$

it is sufficient to solve the equation $\bar{\partial}_b^\Phi v = g$ for $g \in \mathcal{B}_{n-p,r+p}^l(\mathbb{M})$ and this can be done in the same way as in the case of the small degrees, but using the vanishing of the cohomology groups $H^{n-p,p+r}(\mathbb{M})$ for $0 \leq p \leq n - k - r$ and $s_2(q) \leq r \leq n - k$. \square

Assume the horizontal perturbation of the original CR structure on \mathbb{M} is smooth, i.e. Φ is of class \mathcal{C}^∞ , then we can defined on $\widehat{\mathbb{M}}$ the cohomology groups

$$H_{\Phi}^{o,r}(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}_{o,r}^\infty(\widehat{\mathbb{M}}) \mid f_{r,0} = 0, \bar{\partial}_b^\Phi f = 0\} / \bar{\partial}_b^\Phi(\mathcal{C}_{o,r-1}^\infty(\widehat{\mathbb{M}}))$$

and

$$H_{\Phi}^{n,r}(\widehat{\mathbb{M}}) = \{f \in \mathcal{C}_{o,r}^\infty(\widehat{\mathbb{M}}) \mid \bar{\partial}_b^\Phi f = 0\} / \bar{\partial}_b^\Phi(\mathcal{C}_{n,r-1}^\infty(\widehat{\mathbb{M}}))$$

for $1 \leq r \leq n - k$.

Corollary 2.3. *Under the hypotheses of Theorem 2.1, if the sequence $(\mathcal{B}^l(\mathbb{M}), l \in \mathbb{N})$ of anisotropic Banach spaces is such that $\cap_{l \in \mathbb{N}} \mathcal{B}^l(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$ and if the horizontal perturbation of the original CR structure on \mathbb{M} is smooth, there exists a sequence $(\delta_l)_{l \in \mathbb{N}}$ of positive real numbers such that, if $\|\Phi\|_{\theta(l)} < \delta_l$ for each $l \in \mathbb{N}$*

- (i) $H^{p,r-p}(\mathbb{M}) = 0$, for all $1 \leq p \leq r$, implies $H_{\Phi}^{o,r}(\widehat{\mathbb{M}}) = 0$, when $1 \leq r \leq s_1(q)$,
- (ii) $H^{n-p,p+r}(\mathbb{M}) = 0$, for all $0 \leq p \leq n - k - r$, implies $H_{\Phi}^{n,r}(\widehat{\mathbb{M}}) = 0$, when $s_2(q) \leq r \leq n - k$.

Proof. It is a direct consequence of the proof of Theorem 2.1 by the uniqueness of the fixed point of Θ . \square

3 Anisotropic spaces

In the previous section the main theorem is proved under the assumption of the existence of sequences of anisotropic spaces on abstract CR manifolds satisfying good properties with respect to the tangential Cauchy-Riemann operator. We will precise Theorem 2.1 by considering some Sobolev and some Hölder anisotropic spaces for which global homotopy formulas for the tangential Cauchy-Riemann equation with good estimates hold under some geometrical conditions.

In this section $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ denotes an abstract compact CR manifold of class \mathcal{C}^∞ , of real dimension $2n - k$ and CR dimension $n - k$.

Let us define some anisotropic Sobolev spaces of functions:

- $\mathcal{S}^{0,p}(\mathbb{M})$, $1 < p < \infty$, is the set of L_{loc}^p functions on \mathbb{M} .
- $\mathcal{S}^{1,p}(\mathbb{M})$, $1 < p < \infty$, is the set of functions on \mathbb{M} such that $f \in W^{\frac{1}{2},p}(\mathbb{M})$ and $X_{\mathbb{C}}f \in L_{loc}^p(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to M .
- $\mathcal{S}^{l,p}(\mathbb{M})$, $l \geq 2$, $1 < p < \infty$, is the set of functions f such that $Xf \in \mathcal{S}^{l-2,p}(\mathbb{M})$, for all tangent vector fields X to M and $X_{\mathbb{C}}f \in \mathcal{A}^{l-1,p}(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to M .

The sequence $(\mathcal{S}^{l,p}(\mathbb{M}), l \in \mathbb{N})$ is a sequence of anisotropic spaces in the sense of section 2 with $\theta(l) = l + 1$. Moreover $\cap_{l \in \mathbb{N}} \mathcal{S}^{l,p}(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$.

The anisotropic Hölder space of forms $\mathcal{S}_*^{l,p}(\mathbb{M})$, $l \geq 0$, $1 < p < \infty$, is then the space of forms on \mathbb{M} , whose coefficients are in $\mathcal{S}^{l,p}(\mathbb{M})$.

We have now to see if the sequence $(\mathcal{S}^{l,p}(\mathbb{M}), l \in \mathbb{N})$ is adapted to the $\bar{\partial}_b$ -equation for some degree r .

The L^2 theory for \square_b in abstract CR manifolds of arbitrary codimension is developed in [9]. There it is proved that if \mathbb{M} satisfies condition Y(r) the Hodge decomposition theorem holds in degree r , which means that there exist a compact operator $N_b : L_{p,r}^2(\mathbb{M}) \rightarrow \text{Dom}(\square_b)$ and a continuous operator $H_b : L_{p,r}^2(\mathbb{M}) \rightarrow L_{p,r}^2(\mathbb{M})$ such that for any $f \in L_{p,r}^2(\mathbb{M})$

$$f = \bar{\partial}_b \bar{\partial}_b^* N_b f + \bar{\partial}_b^* \bar{\partial}_b N_b f + H_b f \quad (3.1)$$

Moreover H_b vanishes on exact forms and if N_b is also defined on $L_{p,r+1}^2(\mathbb{M})$ then $N_b \bar{\partial}_b = \bar{\partial}_b N_b$.

Therefore if \mathbb{M} satisfy both conditions Y(r) and Y(r+1) then (3.1) becomes an homotopy formula and using the Sobolev and the anisotropic Sobolev estimates in [9] (Theorems 3.3 and Corollary 1.3 (2)) we get the following result:

Proposition 3.1. *If \mathbb{M} is q -concave, the sequence $(\mathcal{S}^{l,p}(\mathbb{M}), l \in \mathbb{N})$ of anisotropic spaces is adapted to the $\bar{\partial}_b$ -equation in degree r for $0 \leq r \leq q - 2$ and $n - k - q + 1 \leq r \leq n - k$.*

Let us define now some anisotropic Hölder spaces of functions:

- $\mathcal{A}^\alpha(\mathbb{M})$, $0 < \alpha < 1$, is the set of continuous functions on \mathbb{M} which are in $\mathcal{C}^{\alpha/2}(\mathbb{M})$.
- $\mathcal{A}^{1+\alpha}(\mathbb{M})$, $0 < \alpha < 1$, is the set of functions f such that $f \in \mathcal{C}^{(1+\alpha)/2}(\mathbb{M})$ and $X_{\mathbb{C}}f \in \mathcal{C}^{\alpha/2}(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to \mathbb{M} . Set

$$\|f\|_{A\alpha} = \|f\|_{(1+\alpha)/2} + \sup_{\|X_{\mathbb{C}}\| \leq 1} \|X_{\mathbb{C}}f\|_{\alpha/2} \quad (3.2)$$

- $\mathcal{A}^{l+\alpha}(\mathbb{M})$, $l \geq 2$, $0 < \alpha < 1$, is the set of functions f of class $\mathcal{C}^{[l/2]}$ such that $Xf \in \mathcal{A}^{l-2+\alpha}(\mathbb{M})$, for all tangent vector fields X to M and $X_{\mathbb{C}}f \in \mathcal{A}^{l-1+\alpha}(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to \mathbb{M} .

The sequence $(\mathcal{A}^{l+\alpha}(\mathbb{M}), l \in \mathbb{N})$ is a sequence of anisotropic spaces in the sense of section 2 with $\theta(l) = l + 1$. Moreover $\cap_{l \in \mathbb{N}} \mathcal{A}^{l+\alpha}(\mathbb{M}) = \mathcal{C}^{\infty}(\mathbb{M})$.

The anisotropic Hölder space of forms $\mathcal{A}_*^{l+\alpha}(\mathbb{M})$, $l \geq 0$, $0 < \alpha < 1$, is then the space of continuous forms on \mathbb{M} , whose coefficients are in $\mathcal{A}^{l+\alpha}(\mathbb{M})$.

It remains to see if the sequence $(\mathcal{A}^{l+\alpha}(\mathbb{M}), l \in \mathbb{N})$ is adapted to the $\bar{\partial}_b$ -equation for some degrees r .

Assume \mathbb{M} is locally embeddable and 1-concave. Then, by Proposition 3.1 in [4], there exist a complex manifold X and a smooth generic embedding $\mathcal{E} : \mathbb{M} \rightarrow M \subset X$ such that M is a smooth compact CR submanifold of X with the CR structure $H_{0,1}M = d\mathcal{E}(H_{0,1}\mathbb{M}) = T_{\mathbb{C}}M \cap T_{0,1}$. If E is a CR vector bundle over \mathbb{M} , by the 1-concavity of \mathbb{M} and after an identification between \mathbb{M} and M , the CR bundle E can be extended to an holomorphic bundle in a neighborhood of M , which we still denote by E . With these notations it follows from [5] that if \mathbb{M} is q -concave, $q \geq 1$, there exist finite dimensional subspaces \mathcal{H}_r of $\mathcal{Z}_{n,r}^{\infty}(\mathbb{M}, E)$, $0 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$, where $\mathcal{H}_0 = \mathcal{Z}_{n,0}^{\infty}(\mathbb{M}, E)$, continuous linear operators

$$A_r : \mathcal{C}_{n,r}^0(\mathbb{M}, E) \rightarrow \mathcal{C}_{n,r-1}^0(\mathbb{M}, E), \quad 1 \leq r \leq q \text{ and } n - k - q + 1 \leq r \leq n - k$$

and continuous linear projections

$$P_r : \mathcal{C}_{n,r}^0(\mathbb{M}, E) \rightarrow \mathcal{C}_{n,r}^0(\mathbb{M}, E), \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k,$$

with

$$\text{Im } P_r = \mathcal{H}_r, \quad 0 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (3.3)$$

and

$$\mathcal{C}_{n,r}^0(M, E) \cap \bar{\partial}_b \mathcal{C}_{n,r-1}^0(M, E) \subseteq \text{Ker } P_r, \quad 1 \leq r \leq q - 1 \text{ and } n - k - q + 1 \leq r \leq n - k, \quad (3.4)$$

such that:

- (i) For all $l \in \mathbb{N}$ and $1 \leq r \leq q$ or $n - k - q + 1 \leq r \leq n - k$,

$$A_r(\mathcal{A}_{n,r}^{l+\alpha}(\mathbb{M}, E)) \subset \mathcal{A}_{n,r-1}^{l+1+\alpha}(\mathbb{M}, E)$$

and A_r is continuous as an operator between $\mathcal{A}_{n,r}^{l+\alpha}(\mathbb{M}, E)$ and $\mathcal{A}_{n,r-1}^{l+1+\alpha}(\mathbb{M}, E)$.

- (ii) For all $0 \leq r \leq q - 1$ or $n - k - q + 1 \leq r \leq n - k$ and $f \in \mathcal{C}_{n,r}^0(\mathbb{M}, E)$ with $\bar{\partial}_b f \in \mathcal{C}_{n,r+1}^0(\mathbb{M}, E)$,

$$f - P_r f = \begin{cases} A_1 \bar{\partial}_b f & \text{if } r = 0, \\ \bar{\partial}_b A_r f + A_{r+1} \bar{\partial}_b f & \text{if } 1 \leq r \leq q - 1 \text{ or } n - k - q + 1 \leq r \leq n - k. \end{cases} \quad (3.5)$$

This implies the following result

Proposition 3.2. *If \mathbb{M} is locally embeddable and q -concave the sequence $\mathcal{A}^{l+\alpha}(\mathbb{M})$, $l \in \mathbb{N}$ of anisotropic spaces is adapted to the $\bar{\partial}_b$ -equation in degree r for $1 \leq r \leq q-1$ and $n-k-q+1 \leq r \leq n-k$*

Finally let us recall the definition of the anisotropic Hölder spaces $\Gamma^{l+\alpha}(\mathbb{M})$ of Folland and Stein.

- $\Gamma^\alpha(\mathbb{M})$, $0 < \alpha < 1$, is the set of continuous functions in \mathbb{M} such that if for every $x_0 \in \mathbb{M}$

$$\sup_{\gamma(\cdot)} \frac{|f(\gamma(t)) - f(x_0)|}{|t|^\alpha} < \infty$$

for any complex tangent curve γ through x_0 .

- $\Gamma^{l+\alpha}(\mathbb{M})$, $l \geq 1$, $0 < \alpha < 1$, is the set of continuous functions in \mathbb{M} such that $X_{\mathbb{C}} f \in \Gamma^{l-1+\alpha}(\mathbb{M})$, for all complex tangent vector fields $X_{\mathbb{C}}$ to \mathbb{M} .

The spaces $\Gamma^{l+\alpha}(\mathbb{M})$ are subspaces of the spaces $\mathcal{A}^{l+\alpha}(\mathbb{M})$.

Note that by Corollary 1.3 (1) in [9] and Section 3 in [5] Propositions 3.1 and 3.2 hold also for the anisotropic Hölder spaces $\Gamma^{l+\alpha}(\mathbb{M})$ of Folland and Stein.

Let us summarize all this in connection with section 2 in the next theorem.

Theorem 3.3. *If \mathbb{M} is q -concave,*

(i) *Theorem 2.1 holds for $\mathcal{B}^l(\mathbb{M}) = \mathcal{S}^{l,p}(\mathbb{M})$ with $s_1(q) = q-2$ and $s_2(q) = n-k-q+1$ in the abstract case,*

(ii) *Theorem 2.1 holds for $\mathcal{B}^l(\mathbb{M}) = \mathcal{A}^{l+\alpha}(\mathbb{M})$ with $s_1(q) = q-1$ and $s_2(q) = n-k-q+1$ when \mathbb{M} is locally embeddable*

(iii) *Theorem 2.1 holds for $\mathcal{B}^l(\mathbb{M}) = \Gamma^{l+\alpha}(\mathbb{M})$ with $s_1(q) = q-2$ in the abstract case and $s_1(q) = q-1$ when \mathbb{M} is locally embeddable, and with $s_2(q) = n-k-q+1$ in both cases.*

Since in all the three cases of Theorem 3.3 we have $\cap_{l \in \mathbb{N}} B_l(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$, Corollary 2.3 becomes

Corollary 3.4. *Let $\mathbb{M} = (\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class \mathcal{C}^∞ , of real dimension $2n-k$ and CR dimension $n-k$, and $\widehat{\mathbb{M}} = (\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ another abstract compact CR manifold such that $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal smooth perturbation of $H_{0,1}\mathbb{M}$. Let $\Phi \in \mathcal{C}_{0,1}^\infty(\mathbb{M}, H_{1,0}\mathbb{M})$ be the differential form which defines the tangential Cauchy-Riemann operator $\bar{\partial}_b^\Phi = \bar{\partial}_b - \Phi \lrcorner \partial_b$ associated to the CR structure $\widehat{H}_{0,1}\mathbb{M}$. Assume \mathbb{M} is q -concave, then there exists a sequence $(\delta_l)_{l \in \mathbb{N}}$ of positive real numbers such that, if $\|\Phi\|_l < \delta_l$ for each $l \in \mathbb{N}$*

(i) *$H^{p,r-p}(\mathbb{M}) = 0$, for all $1 \leq p \leq r$, implies $H_{\Phi}^{0,r}(\widehat{\mathbb{M}}) = 0$, when $1 \leq r \leq q-2$ in the abstract case and also for $r = q-1$ if \mathbb{M} is locally embeddable,*

(ii) *$H^{n-p,r+p}(\mathbb{M}) = 0$, for all $0 \leq p \leq n-k-r$, implies $H_{\Phi}^{n,r}(\widehat{\mathbb{M}}) = 0$, when $n-k-q+1 \leq r \leq n-k$.*

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